

ON THE INFLUENCE OF THE THEIL-LIKE INEQUALITY MEASURE ON THE GROWTH

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ABSTRACT. We set in this paper a coherent theory based on functional empirical processes to consider both the poverty and the inequality indices in one Gaussian field enabling to study the influence of the one on the other. We use the General Poverty Index (*GPI*), that is a class of poverty indices covering the most common ones and a functional class of inequality measure including the Entropy Measure, the Mean Logarithmic Deviation, the different inequality measures of Atkinson, Champernowne, Kolm and Theil called Theil-like Inequality Measures *TLIM*. Our results are given in a unified approach with respect to the two classes instead of their particular elements. We provide the asymptotic laws of the variations of each class over two given periods and the ratio of the variation and derive confidence intervals for them. Although the variances may seem somehow complicated, we provide R codes for their computations and apply the results for the pseudo-panel data for Senegal with simple analysis.

1. INTRODUCTION

In many cases, one has to monitor a specific situation through some risk measure J on some population. The variation of J over time is called growth in case of positive variation and recession alternatively. This growth or recession is not itself sufficient to describe the improvement or deterioration of the situation. Often, the distribution of the underlying variable over the population should also be taken into account in order to check whether the growth concerns a great number of individuals or is rather concentrated on a few number of them.

In the particular case of welfare analysis, one may measure poverty (or richness) with the help of poverty indices J based on the income variable X . Over two periods $s=1$ and $t=2$, we say that we have a

Key words and phrases. Functional empirical process; asymptotic normality; welfare and inequality measure; weak laws, pro and anti-poor growth.

gain against poverty when $\Delta J(s, t) = J(t) - J(s) \leq 0$, or simply a growth against poverty. Before claiming any victory, one must be sure that meanwhile the income did not become more *unequally* distributed, that is the appropriate inequality coefficient I did not decrease. One can achieve this by studying the ratio $R = \Delta J(s, t) / \Delta I(s, t)$.

To make the idea more precise, let us suppose that we are monitoring the poverty scene on some population over the period time $[1, 2]$ and let (X^1, X^2) be the income variable of that population at periods 1 and 2. Let us consider one sample of $n \geq 1$ individuals or households, and observe the income couple $Z_j = (X_j^1, X_j^2)$, $j = 1, \dots, n$. For each period $i \in \{1, 2\}$, we compute the poverty measure $J_n(i)$ and the inequality measure $I_n(i)$. We draw the attention of the reader that we consider here classes of measures both for poverty and inequality rather than specific ones. This leads to very general results but requires extended notation.

For poverty, we consider the Generalized Poverty Index (GPI) introduced by Lo *and al.* [30] as an attempt to gather a large class of poverty measures reviewed in Zheng [35] defined as follows for period i ,

(1.1)

$$J_n(i) = \frac{A(Q_n(i), n, Z(i))}{nB(Q_n(i))} \sum_{j=1}^{Q_n(i)} w(\mu_1 n + \mu_2 Q_n(i) - \mu_3 j + \mu_4) d\left(\frac{Z(i) - Y_{j,n}}{Z(i)}\right)$$

where $B(Q_n) = \sum_{j=1}^n w(j)$, μ_1, μ_2, μ_3 and μ_4 are constants, $A(u, v, s)$, $w(t)$, and $d(y)$ are mesurable functions of $(u, v, s) \in \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+^*$, $t \in \mathbb{R}_+^*$, and $y \in (0, 1)$. By particularizing the functions A and w and by giving fixed values to the μ'_i s, we may find almost all the available indices, as we will do it later on. *In the sequel, (1.1) will be called a poverty index (indices in the plural) or simply a poverty measure according to the economists terminology.*

This class includes the most popular indices such as those of Sen ([32]), Kakwani ([22]), Shorrocks ([1]), Clark-Hemming-Ulph ([8]), Foster-Greer-Thorbecke ([15]), etc. See Lo ([24]) for a review of the GPI. From the works of many authors ([28], [29] for instance), $J_n(i)$ is an asymptotically sufficient estimate of the exact poverty measure

$$(1.2) \quad J(i) = \int_0^{Z(i)} L(y, G_i) d\left(\frac{Z(i) - y}{Z(i)}\right) dG_i(y)$$

where G_i is the distribution function of X^i ($i = 1, 2$), and L is some weight function.

As for the inequality measure, we use this Theil-like family, where we gathered the Generalized Entropy Measure, the Mean Logarithmic Deviation ([14], [34], [9]), the different inequality measures of Atkinson ([3]), Champernowne ([7]) and Kolm ([23]) in the following form:

$$(1.3) \quad I_n(i) = \tau \left(\frac{1}{h_1(\mu_n(i))} \frac{1}{n} \sum_{j=1}^n h(X_j^i) - h_2(\mu_n(i)) \right)$$

where $\mu_n(i) = \frac{1}{n} \sum_{j=1}^n X_j^i$ denotes the empirical mean while h , h_1 , h_2 , and τ are measurable functions.

The inequality measures mentioned above are derived from (1.3) with the particular values of α, τ, h, h_1 and h_2 as described below for all $s > 0$:

(a) Generalized Entropy

$$\alpha \neq 0, \alpha \neq 1, \tau(s) = \frac{s-1}{\alpha(\alpha-1)}, h(s) = h_1(s) = s^\alpha, h_2(s) \equiv 0;$$

(b) Theil's measure:

$$\tau(s) = s, h(s) = s \log(s), h_1(s) = s, h_2(s) = \log(s);$$

(c) Mean Logarithmic Deviation

$$\tau(s) = s, h(s) = h_2(s) = \log(s^{-1}), h_1(s) \equiv 1;$$

(d) Atkinson's measure:

$$\alpha < 1 \text{ and } \alpha \neq 0, \tau(s) = 1 - s^{1/\alpha}, h(s) = h_1(s) = s^\alpha, h_2(s) \equiv 0;$$

(e) Champernowne's measure:

$$\tau(s) = 1 - \exp(s), h(s) = h_2(s) = \log(s), h_1(s) \equiv 1;$$

(f) Kolm's measure:

$$\alpha > 0, \tau(s) = \frac{1}{\alpha} \log(s), h(s) = h_1(s) = \exp(-\alpha s), h_2(s) \equiv 0.$$

We will see below that $I_n(i)$ converges to the exact inequality measure

$$(1.4) \quad I(i) = \tau \left(\frac{1}{h_1(\mu(i))} \int_{\mathbb{R}} h(x) dG_i(x) - h_2(\mu(i)) \right)$$

where $\mu(i) = \mathbb{E}(X^i)$ is the mathematical expectation of X^i that we suppose finite here.

The motivations stated above lead to the study of the behavior of $(\Delta J_n(s, t), \Delta I_n(s, t))$ as an estimate of the unknown value of $(\Delta J(s, t), \Delta I(s, t))$. Precisely a confidence interval of

$$R(s, t) = \frac{\Delta J(s, t)}{\Delta I(s, t)}$$

will be an appropriate set of tools for the study of the influence of each measure on the other.

To achieve our goal we need a coherent asymptotic theory allowing the handling of longitudinal data as it is the case here and a stochastic process approach leading to asymptotic sub results with the help of the continuity mapping theorem.

We find that the functional empirical process, in the modern setting of weak convergence theory, provides *that* coherent asymptotic theory.

Indeed, we use bidimensional functional empirical processes \mathbb{G}_n associated with Z_1, Z_2, \dots, Z_n and its stochastic Gaussian limit \mathbb{G} to entirely describe the asymptotic behaviour of $(\Delta J_n(s, t), \Delta I_n(s, t))$ in the Gaussian field of \mathbb{G} and then find the law of $R_n(s, t) = \Delta J_n(s, t) / \Delta I_n(s, t)$ as our best achievements.

The remainder of the paper is organized as follows. In Section 2, we remind key definitions and properties for functional empirical processes, and we state the asymptotic representation of the GPI of Lo stated in Theorem (1) that will be used later on. In Section 3, we give our main results and make some commentaries and data driven applications to Senegalese pseudo-panel data are considered. In section 4, proof of the theorems. The paper is ended by concluding remarks in Section 5.

2. FUNCTIONAL EMPIRICAL PROCESS AND REPRESENTATION OF GPI

2.1. A brief reminder on Functional Empirical Processes. Let Z_1, Z_2, \dots, Z_n a sequence independent and identically distributed of random elements with values in some metric space (S, d) . Given a

collection \mathcal{F} of measurable functions $f : S \rightarrow \mathbb{R}$, the functional empirical process (*FEP*) is defined by:

$$\forall f \in \mathcal{F}, \mathbb{G}_n(f) = \frac{1}{\sqrt{n}} \sum_{j=1}^n (f(Z_j) - \mathbb{E}f(Z_j)).$$

This process is widely studied in van der Waart [2] for instance. It is directly seen that whenever $\mathbb{E}(f(Z)^2) < \infty$, one has $\frac{1}{n} \sum_{j=1}^n f(Z_j) \rightarrow \mathbb{P}(f) = \mathbb{E}(f(Z))$ *a.s.* and $\mathbb{G}_n(f) \rightarrow \mathcal{N}(0, \sigma_f^2)$ where

$$(2.1) \quad \sigma_f^2 = \mathbb{E}((f(Z) - \mathbb{P}(f))^2),$$

as consequences of the real Law of Large Numbers (*LLN*) and the real Central Limit Theorem (*CLT*).

When using the *FEP*, we are often interested in uniform *LLN*'s and weak limits of the *FEP* considered as stochastic processes. This gives the so important results on Glivenko-Cantelli classes and Donsker ones. Let us define them here (for more details see [2]).

Definition 1. A class $\mathcal{F} \subset L_1(\mathbb{P})$ is called a *Glivenko-Cantelli class* for \mathbb{P} , if

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{j=1}^n (f(Y_j) - \mathbb{E}f(Y_j)) \right\|_{\mathcal{F}} = \lim_{n \rightarrow \infty} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{j=1}^n (f(Y_j) - \mathbb{E}f(Y_j)) \right| = 0 \text{ as.}$$

Definition 2. A class $\mathcal{F} \subset L_2(\mathbb{P})$ is called a *Donsker class* for \mathbb{P} , or *\mathbb{P} -Donsker class* if $\{\mathbb{G}_n(f); f \in \mathcal{F}\}$ converge in $l^\infty(\mathcal{F})$ to a centered Gaussian process $\{\mathbb{G}(f); f \in \mathcal{F}\}$ with covariance function

$$\mathbb{E}(\mathbb{G}(f) \mathbb{G}(g)) = \int_{\mathbb{R}} (f(y) - \mathbb{E}(f(Y)))(g(y) - \mathbb{E}(g(Y))) d\mathbb{P}_Y(y); \forall f, g \in \mathcal{F}.$$

Remark 1. When $S = \mathbb{R}$ and $\mathcal{F} = \{\mathbb{I}_{(-\infty, x]}; x \in \mathbb{R}\}$, \mathbb{G}_n is called *real empirical process* and is often denoted by α_n .

In this paper, we only use finite-dimensional forms of the FEP, that is $(\mathbb{G}_n(f_i), i = 1, \dots, k)$. And then, any family $\{f_i, i = 1, \dots, k\}$ of measurable functions satisfying (2.1), is a Glivenko-Cantelli and a Donsker class, and hence

$$(\mathbb{G}_n(f_i), i = 1, \dots, k) \xrightarrow{d} (\mathbb{G}(f_1), \mathbb{G}(f_2), \dots, \mathbb{G}(f_k))$$

where \mathbb{G} is the Gaussian process, defined in Definition 2.

We will make use of the linearity property of both \mathbb{G}_n and \mathbb{G} . Let f_1, \dots, f_k measurable functions satisfying (2.1) and $a_i \in \mathbb{R}, i = 1, \dots, k$, then

$$\sum_{j=1}^k a_j \mathbb{G}_n(f_j) = \mathbb{G}_n\left(\sum_{j=1}^k a_j f_j\right) \xrightarrow{d} \mathbb{G}\left(\sum_{j=1}^k a_j f_j\right).$$

The materials defined here, when used in a smart way, lead to a simple handling the problem tackled here.

2.2. Representation of the GPI. In this paper, we use the GPI in unified approach that leads to an asymptotic representation for a large class of indices. For this, let it be the following hypotheses. Different kinds of conditions are needed.

First we consider this threshold condition:

(H1) There exist $\beta > 0$ and $0 < \xi < 1$ such that,

$$0 < \beta < G(Z) < \xi < 1.$$

Next we have form conditions (on the indices):

(H2a) There exist a function $h(p, q)$ where $(p, q) \in \mathbb{N}^2$ and a function $c(s, t)$ where $(s, t) \in (0, 1)^2$ such that, when $n \rightarrow +\infty$,

$$\max_{1 \leq j \leq Q} \left| A(n, Q) h^{-1}(n, Q) w(\mu_1 n + \mu_2 Q - \mu_3 j + \mu_4) - c(Q/n, j/n) \right| = o_{\mathbb{P}}(n^{-1/2});$$

(H2b) There exists a function $\pi(s, t)$ with $(s, t) \in \mathbb{R}^2$ such that, when $n \rightarrow +\infty$,

$$\max_{1 \leq j \leq Q} \left| w(j) h^{-1}(n, Q) - \frac{1}{n} \pi(Q/n, j/n) \right| = o_{\mathbb{P}}(n^{-3/2}).$$

Further we need regularity conditions on c and π :

(H3) The functions $c(\cdot)$ and $\pi(\cdot)$ have uniformly continuous partial derivatives, that is

$$\lim_{(k,l) \rightarrow (0,0)} \sup_{(x,y) \in (0,1)^2} \left| \frac{\partial c}{\partial y}(x+l, y+k) - \frac{\partial c}{\partial y}(x, y) \right| = 0$$

and

$$\lim_{(k,l) \rightarrow (0,0)} \sup_{\beta \leq x \leq \xi, y \in (0,1)} \left| \frac{\partial c}{\partial x}(x+l, y+k) - \frac{\partial c}{\partial x}(x, y) \right| = 0;$$

(H4) The functions $y \rightarrow \frac{\partial c}{\partial y}(x, y)$ and $y \rightarrow \frac{\partial \pi}{\partial y}(x, y)$ are monotonous.

(H5) The distribution function G is increasing.

(H6) There exist $H_0 > 0$ and $H_\infty < +\infty$ such that

$$H_0 < H_c(G) = \int_0^{+\infty} c(G(Z), G(y))\gamma(y)dG(y) < H_\infty,$$

and

$$H_0 < H_\pi(G) = \int_0^{+\infty} \pi(G(Z), G(y))e(y)dG(y) < H_\infty$$

where

$$\gamma(x) = d\left(\frac{Z-x}{Z}\right) \mathbb{I}_{(x \leq Z)} \text{ and } e(x) = \mathbb{I}_{(x \leq Z)} \text{ for } x \in \mathbb{R}.$$

Finally define

$$J(G) = H_c(G)/H_\pi(G),$$

$$g(\cdot) = H_\pi^{-1}(G)g_c(\cdot) - H_c(G)H_\pi^{-2}(G)g_\pi(\cdot) + K(G)e(\cdot),$$

with

$$g_c(\cdot) = c(G(Z), G(\cdot))\gamma(\cdot), \quad g_\pi(\cdot) = \pi(G(Z), G(\cdot))e(\cdot),$$

$$K(G) = H_\pi^{-1}(G)K_c(G) - H_c(G)H_\pi^{-2}(G)K_\pi(G)$$

where

$$K_c(G) = \int_0^1 \frac{\partial c}{\partial x}(G(Z), s)\gamma(G^{-1}(s))ds, \quad K_\pi(G) = \int_0^1 \frac{\partial \pi}{\partial x}(G(Z), s)e(G^{-1}(s))ds,$$

$$\nu(\cdot) = H_\pi^{-1}(G)\nu_c(\cdot) - H_c(G)H_\pi^{-2}(G)\nu_\pi(\cdot),$$

and

$$\nu_c(\cdot) = \frac{\partial c}{\partial y}(G(Z), G(\cdot))\gamma(\cdot), \quad \nu_\pi(\cdot) = \frac{\partial \pi}{\partial y}(G(Z), G(\cdot))e(\cdot);$$

$$\alpha_n(g) = \frac{1}{\sqrt{n}} \sum_{j=1}^n g(Y_j) - \mathbb{E}g(Y_j)$$

is the functional empirical process and

$$\beta_n(\nu) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \{G_n(Y_j) - G(Y_j)\} \nu(Y_j)$$

is the reduced process of Sall et LO (see [27]).

The representation results of [27] for the GPI is the following.

Theorem 1. *Suppose that (H1)-(H6) are true, then we have the following representation*

$$(R) \quad \sqrt{n}(J_n(G) - J(G)) = \alpha_n(g) + \beta_n(\nu) + o_{\mathbb{P}}(1).$$

Although these conditions may appear complicated, they are simple to check in real cases with the popular poverty measures. We will see this in Section 3.

We are going to state our mains results.

3. RESULTS AND COMMENTARIES

3.1. Notations. Let us consider the following Renyi representations. Let $\{U_j\}_{j=1,\dots,n}$ and $\{V_j\}_{j=1,\dots,n}$ two sequences of independent uniform rv's on $I = (0, 1)$. Then we have the representation, meant as equality in distribution:

$$X_j^1 = G_1^{-1}(U_j) \quad \text{and} \quad X_j^2 = G_2^{-1}(V_j), j \in \{1, \dots, n\}$$

where G_i^{-1} is the generalized inverse of G_i . We suppose that G_i is continuous. The copula associated with the couple (X^1, X^2) is defined by

$$C(u, v) = G_{1,2}(G_1^{-1}(u), G_2^{-1}(v)), \forall (u, v) \in I^2$$

where $G_{1,2}$ is the joint distribution function of (X^1, X^2) .

Next we consider the bidimensional functional empirical process based on $\{(U_j, V_j)\}_{j=1,\dots,n}$, for some Donsker class \mathcal{F} :

$$\forall f \in \mathcal{F}, \quad \mathbb{G}_n(f) = \frac{1}{\sqrt{n}} \sum_{j=1}^n (f(U_j, V_j) - \mathbb{P}_{(U,V)}(f));$$

and the limiting centred Gaussian stochastic process \mathbb{G} defined by its variance-covariance function, for $(f, g) \in \mathcal{F}^2$:

$$\mathbb{E}(\mathbb{G}(f) \mathbb{G}(g)) = \int_{I^2} (f(u, v) - \mathbb{P}_{(U,V)}(f)) (g(u, v) - \mathbb{P}_{(U,V)}(g)) dC(u, v)$$

where

$$\mathbb{P}_{(U,V)}(f) = \mathbb{E}(f(U, V)) = \int_{I^2} f(u, v) dC(u, v).$$

Now we introduce the following notation based of the functions τ, h, h_1, h_2 of (1.3) and on the functions g and ν of Theorem 1. The subscript i refers to the periods. The series of notations are about the variation of the inequality measures and are listed below. Let first

$$B_n(i) = \frac{1}{n} \sum_{j=1}^n h(X_j^i), \quad B(i) = \int_{\mathbb{R}} h(x) dG_i(x);$$

and next, for all $(u, v) \in I^2$,

$$\tilde{f}_i(u, v) = G_i^{-1} \circ \Pi_i(u, v)$$

where Π_i is the i^{th} projections of $(0, 1)^2$,

$$f_{i,h}(u, v) = h \circ \tilde{f}_i(u, v).$$

And finally

$$F_{i,I}^*(u, v) = K_i \left(\frac{1}{h_1(\mu(i))} f_{i,h}(u, v) - \left(\frac{B(i)h'_1(\mu(i))}{h_1^2(\mu(i))} + h'_2(\mu(i)) \right) \tilde{f}_i(u, v) \right)$$

where $K_i = \tau' \left(\frac{B(i)}{h_1(\mu(i))} - h_2(\mu(i)) \right)$ and $F_I^*(u, v) = F_{2,I}^*(u, v) - F_{1,I}^*(u, v)$.

For our results on the variation of the GPI, we need the functions g_i and ν_i provided by the representation of the Theorem 1. Put accordingly with these functions:

$$g_i(x) = c(G_i(x)) q_i(x) \text{ and } \nu_i(s) = c'(s) q_i(G_i^{-1}(s)).$$

We define for all $(u, v) \in I^2$

$$f_{i,s}(u, v) = \Pi_i(\mathbb{I}_{(o,s)}(u), \mathbb{I}_{(o,s)}(v)),$$

$$F_{i,J}^*(u, v) = g_i \circ \tilde{f}_i(u, v) = g_i \circ G_i^{-1} \circ \Pi_i(u, v),$$

and

$$F_J^*(u, v) = F_{2,J}^*(u, v) - F_{1,J}^*(u, v).$$

3.2. Main Theorems. We are now able to states our theorems. The first concerns the variation of the inequality measure.

Theorem 2. *Let $\mu(i)$ finite for $i = 1, 2$. Let $\mathbb{P}_{(U,V)}(F_I^{\star 2}) < \infty$, then we have the following convergence as $n \rightarrow \infty$*

$$\sqrt{n}(\Delta I_n(1, 2) - \Delta I(1, 2)) \rightarrow_d \mathcal{N}(0, \Gamma_I(1, 2))$$

where \rightarrow_d stands for the convergence in distribution and

$$\Gamma_I(1, 2) = \int_{I^2} (F_I^{\star}(u, v) - \mathbb{P}_{(U,V)}(F_I^{\star}))^2 dC(u, v).$$

The second concerns the variation of the GPI.

Theorem 3. *Let $\mu(i)$ finite for $i = 1, 2$ and let each h_i continuously differentiable at each $\mu(i)$, $i = 1, 2$. Suppose that $\mathbb{P}_{(U,V)}((f_{1,s})^2)$, $\mathbb{P}_{(U,V)}((f_{2,s})^2)$ and $\mathbb{P}_{(U,V)}(F_J^{\star 2})$ are finite. Then*

$$\sqrt{n}(\Delta J_n(1, 2) - \Delta J(1, 2)) \rightarrow_d \mathbb{G}(F_J^{\star}) + \int_I (\mathbb{G}(f_{2,s}) \nu_2(s) - \mathbb{G}(f_{1,s}) \nu_1(s)) ds$$

which is a centered Gaussian process of variance-covariance function:

$$\Gamma_J(1, 2) = \Gamma_1(1, 2) + \Gamma_2(1, 2) + 2\Gamma_3(1, 2)$$

where

$$\begin{aligned} \Gamma_1(1, 2) &= \int_{I^2} (F_J^{\star}(u, v) - \mathbb{P}_{(U,V)}(F_J^{\star}))^2 dC(u, v); \\ \Gamma_2(1, 2) &= \gamma_1 - 2\gamma_2 + \gamma_3 \end{aligned}$$

with

$$\begin{aligned} \gamma_1 &= \int_{I^2} \nu_2(s) \nu_2(t) (\min\{s, t\} - st) ds dt, \\ \gamma_2 &= \int_{I^2} \nu_2(s) \nu_1(t) (C(t, s) - st) ds dt, \\ \gamma_3 &= \int_{I^2} \nu_1(s) \nu_1(t) (\min\{s, t\} - st) ds dt; \end{aligned}$$

and

$$\begin{aligned} \Gamma_3(1, 2) &= \int_I \left\{ \nu_2(s) \int_{(0,1) \times (0,s)} F_J^{\star}(u, v) dC(u, v) - \nu_1(s) \int_{(0,s) \times (0,1)} F_J^{\star}(u, v) dC(u, v) \right\} ds \\ &\quad - \mathbb{P}_{(U,V)}(F_J^{\star}) \int_I s (\nu_2(s) - \nu_1(s)) ds. \end{aligned}$$

Thus last one handles the ratio of the two variations.

Theorem 4. *Supposing that the above mentioned hypotheses are true, then*

$$(\sqrt{n}(\Delta J_n(1, 2) - \Delta J(1, 2)), \sqrt{n}(\Delta I_n(1, 2) - \Delta I(1, 2)))^t \xrightarrow{d} \mathcal{N}_2(0, \Sigma),$$

with

$$\Sigma = \begin{pmatrix} \Gamma_J(1, 2) & \Gamma_{I,J}(1, 2) \\ \Gamma_{I,J}(1, 2) & \Gamma_I(1, 2) \end{pmatrix}$$

and

$$\begin{aligned} \Gamma_{I,J}(1, 2) &= \int_{I^2} (F_I^*(u, v) - \mathbb{P}_{(U,V)}(F_I^*)) (F_J^*(u, v) - \mathbb{P}_{(U,V)}(F_J^*)) dC(u, v) \\ &+ \int_I \left\{ \nu_2(s) \int_{(0,1) \times (0,s)} F_I^*(u, v) dC(u, v) - \nu_1(s) \int_{(0,s) \times (0,1)} F_I^*(u, v) dC(u, v) \right\} ds \\ &\quad - \mathbb{P}_{(U,V)}(F_I^*) \int_I s (\nu_2(s) - \nu_1(s)) ds. \end{aligned}$$

In this case, let

$$R = \frac{\Delta J(1, 2)}{\Delta I(1, 2)}, \quad a = \frac{1}{\Delta I(1, 2)} \quad \text{and} \quad b = \frac{\Delta J(1, 2)}{(\Delta I(1, 2))^2},$$

then we have

$\sqrt{n} \{R_n(1, 2) - R(1, 2)\}$ tends to a functional Gaussian process

$$a \left(\mathbb{G}(F_J^*) + \int_I (\nu_2(s) \mathbb{G}(f_{2,s}) - \nu_1(s) \mathbb{G}(f_{1,s})) ds \right) - b \mathbb{G}(F_I^*);$$

of covariance function

$$\Gamma(1, 2) = a^2 \Gamma_J(1, 2) + b^2 \Gamma_I(1, 2) - 2ab \Gamma_{I,J}(1, 2).$$

3.3. Commentaries and applications. First of all, the results covers so many poverty measures and inequality indices. This explains why the notation seem heavy. Secondly, the variances of the limiting Gaussian processes seem also somehow tricky. But all of them are easily handled by modern computation means. We are going to particularise our results for famous measures and provide easy software codes for the computations.

3.4. Representation of some poverty indices. We may easily find the functions g and ν for the most common members of the GPI family (See [16], [27]) as listed below.

| Mesure | g | ν |
|------------------|--|---|
| Shorrocks | $2(1 - G(y)) \left(\frac{Z-y}{Z}\right) \mathbb{I}_{(y \leq Z)}$ | $-2 \left(\frac{Z-y}{Z}\right) \mathbb{I}_{(y \leq Z)}$ |
| Thon | $2(1 - G(y)) \left(\frac{Z-y}{Z}\right) \mathbb{I}_{(y \leq Z)}$ | $-2 \left(\frac{Z-y}{Z}\right) \mathbb{I}_{(y \leq Z)}$ |
| Sen | g_s | ν_s |
| Kakwani | g_k | ν_k |

where

$$g_s(y) = \left\{ 2 \left[\left(1 - \frac{G(y)}{G(Z)}\right) \left(\frac{Z-y}{Z}\right) - \left(\frac{G(y)}{G(Z)}\right) \left(\frac{J_s(G)}{G(Z)}\right) \right] + K_s(G) \right\} \mathbb{I}_{(y \leq Z)},$$

and

$$\nu_s(y) = -\frac{2}{G(Z)} \left[\left(\frac{Z-y}{Z}\right) + \frac{J_s(G)}{G(Z)} \right] \mathbb{I}_{(y \leq Z)}.$$

with

$$J_s(G) = 2 \int_0^{G(Z)} \left(1 - \frac{s}{G(Z)}\right) \left(\frac{Z - G^{-1}(s)}{Z}\right) ds,$$

$$K_s(G) = 2 \left(1 - \frac{1}{ZG(Z)} \int_0^{G(Z)} G^{-1}(s) ds\right) + \frac{J_s(G)}{G(Z)}.$$

And

$$g_k(y) = \left\{ (k+1) \left[\left(1 - \frac{G(y)}{G(Z)}\right)^k \left(\frac{Z-y}{Z}\right) - \frac{J_k(G)}{G(Z)} \left(\frac{G(y)}{G(Z)}\right)^k \right] + K_k(G) \right\} \mathbb{I}_{(y \leq Z)},$$

$$\nu_k(y) = -\frac{k(k+1)}{G(Z)} \left[\left(1 - \frac{G(y)}{G(Z)}\right)^{k-1} \left(\frac{Z-y}{Z}\right) + \frac{J_k(G)}{G(Z)} \left(\frac{G(y)}{G(Z)}\right)^{k-1} \right] \mathbb{I}_{(y \leq Z)}$$

where

$$J_k(G) = (k+1) \int_0^{G(Z)} \left(1 - \frac{s}{G(Z)}\right)^k \left(\frac{Z - G^{-1}(s)}{Z}\right) ds,$$

and

$$K_k(G) = \frac{k(k+1)}{G(Z)} \int_0^{G(Z)} \left(1 - \frac{s}{G(Z)}\right)^{k-1} \left(\frac{Z - G^{-1}(s)}{Z}\right) ds + \frac{J_k(G)}{G(Z)}.$$

Notice that the functions are index with k for the Kakwani measure. For the FGT measure of index α , we have that $\nu = 0$ and

$$g(x) = \max(0, (Z - x)/Z)^\alpha.$$

3.5. Datadriven applications and variance computations.

3.5.1. *Variance computations for Senegalese data.* We apply our results to Senegalese data. We do not really have longitudinal data. So we have constructed pseudo-panel data of size $n = 116$, from two surveys: ESAM II conducted from 2001 to 2002 and EPS from 2005 to 2006. We get two series X^1 and X^2 . We present below the values of $\Gamma_I(1, 2)$ denoted here $\gamma(1)$, $\Gamma_J(1, 2)$ denoted here $\gamma(2)$ and $\Gamma(1, 2)$ denoted here $\gamma(3)$.

When constructing pseudo-panel data, we get small sizes like $n=116$ here. We use these sizes to compute the asymptotic variances in our results with nonparametric methods. In real contexts, we should use high sizes comparable to those of the real databases, that is around ten thousands, like in the Senegalese case. Nevertheless, we back on medium sizes, for instance $n=696$, which give very accurate confidence intervals as shown in the tables below.

Before we present the outcomes, let us say some words on the packages. We provide different R script files at:

<http://www.ufrsat.org/lerstad/resources/mergslo01.zip>

The user should already have his data in two files *data1.txt* and *data2.txt*. The first script file named after *gamma_mergslo1.dat* provides the values of $\gamma(1)$, $\gamma(2)$ and $\gamma(3)$ for the FGT measure for $\alpha = 0, 1, 2$ and for the six inequality measures used here. The second script file named as *gamma_mergslo2.dat* performs the same for the Shorrocks measure. Lastly, *gamma_mergslo3.dat* concerns the kakwani measures. Unless the user uploads new *data1.txt* and *data2.txt* files, the outcomes should be the same as those presented in the Appendix.

3.5.2. *Analysis.* First of all, we find that, at an asymptotical level, all our inequality measures and poverty indices used here have decreased. When inspecting the asymptotic variance, we see that for the poverty indice, the FGT and the Kakwani classes respectively for $\alpha = 1$, $\alpha = 2$ and $k = 1$ and $= 2$ have the minimum variance, specially for $\alpha = 2$ and $k = 2$. This advocates for the use of the Kakwani and the FGT measures for poverty reduction evaluation. As for the inequality approach, it seems that Atkinson measure ATK(0.5) has the minimum variance

and then is recommended.

As for the ratio of the poverty index over the inequality measure, we have a dependance of over 50% for the following couples:

(SHOR, GE(0.5)) [75.13%], (SHOR, THEIL) [66.19%], (SHOR MLD) [82.29%], (SHOR ATK(0.5)) [153.06%], (SHOR ATK(-0.5)) [68.37%], (SHOR CHAMP) [88.39%], (SEN, MLD) [57.84%], (SEN, CHAMP) [61.63%], (KAK(2), GE(0.5)) [51.06%], (KAK(2), MLD) [51.06%] (KAK(2), CHAMP) [60.07%], (FGT(1), CHAMP)[54.33%].

The maximum ratio 3.024 is attained for the FGT (0) and Atkinson (0.5). Based on these data, and on the confidence intervals in Table 6 , we would report at least of 46.43% for these two measures and conclude that gain over poverty in Senegal between this two periods is significantly *pro-poor*. We would have worked with all couples with a ratio over 50% to have the same conclusion.

The present analysis should be developped in a separated paper research since this one was devoted to theoritical basis. We plan to apply at a regional basis, that is for the countries of the UEMOA in West Africa.

We finish by the proofs that may be skipped by non mathematician readers.

4. PROOFS OF THE THEOREMS

Theorem 2.

By using the delta-method, we have for all $i \in \{1, 2\}$:

$$\begin{aligned} \sqrt{n} \{h_1(\mu_n(i)) - h_1(\mu(i))\} &= h'_1(\mu(i)) \sqrt{n} (\mu_n(i) - \mu(i)) + o_p(1) \\ &= h'_1(\mu(i)) \frac{1}{\sqrt{n}} \sum_{j=1}^n (X_j^i - \mathbb{E}(X_j^i)) + o_p(1) \\ &= h'_1(\mu(i)) \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(\tilde{f}_i(U_j, V_j) - \mathbb{P}_{(U,V)}(\tilde{f}_i) \right) + o_p(1) \\ &= h'_1(\mu(i)) \mathbb{G}_n(\tilde{f}_i) + o_p(1). \end{aligned}$$

Then

$$(4.1) \quad \sqrt{n} \{h_1(\mu_n(i)) - h_1(\mu(i))\} = \mathbb{G}_n \left(h'_1(\mu(i)) \tilde{f}_i \right) + o_p(1).$$

Similarly, we have

$$(4.2) \quad \sqrt{n} \{h_2(\mu_n(i)) - h_2(\mu(i))\} = \mathbb{G}_n \left(h'_2(\mu(i)) \tilde{f}_i \right) + o_p(1).$$

From this and (3.1), we have

$$\begin{aligned} \sqrt{n} \{B_n(i) - B(i)\} &= \frac{1}{\sqrt{n}} \sum_{j=1}^n (h(X_j^i) - \mathbb{E}(h(X_j^i))) \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^n (f_{i,h}(U_j, V_j) - \mathbb{P}_{(U,V)}(f_{i,h})); \end{aligned}$$

and then

$$(4.3) \quad \sqrt{n} \{B_n(i) - B(i)\} = \mathbb{G}_n(f_{i,h}).$$

Further

$$\begin{aligned} \sqrt{n} \{I_n(i) - I(i)\} &= \sqrt{n} \left\{ \tau \left(\frac{B_n(i)}{h_1(\mu_n(i))} - h_2(\mu_n(i)) \right) - \tau \left(\frac{B(i)}{h_1(\mu(i))} + h_2(\mu(i)) \right) \right\} \\ &= K_i \sqrt{n} \left\{ \frac{B_n(i)}{h_1(\mu_n(i))} - h_2(\mu_n(i)) - \frac{B(i)}{h_1(\mu(i))} + h_2(\mu(i)) \right\} + o_p(1). \end{aligned}$$

But

$$\begin{aligned} \sqrt{n} \left\{ \frac{B_n(i)}{h_1(\mu_n(i))} - h_2(\mu_n(i)) - \frac{B(i)}{h_1(\mu(i))} + h_2(\mu(i)) \right\} &= \frac{\sqrt{n} \{B_n(i) - B(i)\}}{h_1(\mu_n(i))} \\ &\quad - \left(\frac{B(i) h'_1(\mu(i))}{h_1(\mu(i)) h_1(\mu_n(i))} + h'_2(\mu(i)) \right) \sqrt{n} \{\mu_n(i) - \mu(i)\} + o_p(1) \\ &= \frac{1}{h_1(\mu_n(i))} \mathbb{G}_n(f_{i,h}) - \left(\frac{B(i) h'_1(\mu(i))}{h_1(\mu(i)) h_1(\mu_n(i))} + h'_2(\mu(i)) \right) \mathbb{G}_n(\tilde{f}_i) + o_p(1) \\ &= \mathbb{G}_n \left(\frac{1}{h_1(\mu(i))} f_{i,h} - \left(\frac{B(i) h'_1(\mu(i))}{h_1^2(\mu(i))} + h'_2(\mu(i)) \right) \tilde{f}_i \right) + o_p(1). \end{aligned}$$

Thus

$$\sqrt{n} \{I_n(i) - I(i)\} = K_i \mathbb{G}_n \left(\frac{1}{h_1(\mu(i))} f_{i,h} - \left(\frac{B(i) h'_1(\mu(i))}{h_1^2(\mu(i))} + h'_2(\mu(i)) \right) \tilde{f}_i \right) + o_p(1),$$

that is

$$(4.4) \quad \sqrt{n} \{I_n(i) - I(i)\} = \mathbb{G}_n(F_{i,I}^*) + o_p(1).$$

Finally using the linearity of the FEP, we get

$$\begin{aligned}\sqrt{n} \{ \Delta I_n(1, 2) - \Delta I(1, 2) \} &= \sqrt{n} \{ I_n(2) - I(2) \} - \sqrt{n} \{ I_n(1) - I(1) \} \\ &= \mathbb{G}_n(F_{2,I}^*) - \mathbb{G}_n(F_{1,I}^*) + o_p(1) \\ &= \mathbb{G}_n(F_{2,I}^* - F_{1,I}^*) + o_p(1).\end{aligned}$$

and conclude by

$$(4.5) \quad \sqrt{n} \{ \Delta I_n(1, 2) - \Delta I(1, 2) \} = \mathbb{G}_n(F_I^*) + o_p(1)$$

and

$$\Gamma_I(1, 2) = \mathbb{E} \left(\mathbb{G}(F_I^*)^2 \right) = \int_{I^2} \left(F_I^*(u, v) - \mathbb{P}_{(U,V)}(F_I^*) \right)^2 dC(u, v).$$

Proof of Theorem 3. We have

$$J_n(i) = \frac{1}{n} \sum_{j=1}^n c \left(G_n^i(X_{j,n}^i) \right) q_i(X_{j,n}^i)$$

and then

$$\begin{aligned}\sqrt{n} \{ J_n(i) - J(i) \} &= \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(g_i(X_{j,n}^i) - \mathbb{E} g_i(X_{j,n}^i) \right) + \int_I \alpha_n(s) \nu_i(s) ds + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(g_i \circ G_i^{-1} \circ \Pi_i(U_{j,n}, V_{j,n}) - \mathbb{E} g_i \circ G_i^{-1} \circ \Pi_i(U_{j,n}, V_{j,n}) \right) \\ &+ \int_I \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(\Pi_i(\mathbb{I}_{(0,s)}(U_{j,n}), \mathbb{I}_{(0,s)}(V_{j,n})) - \mathbb{E} \Pi_i(\mathbb{I}_{(0,s)}(U_{j,n}), \mathbb{I}_{(0,s)}(V_{j,n})) \right) \nu_i(s) ds + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(F_{i,J}^*(U_{j,n}, V_{j,n}) - \mathbb{P}_{(U,V)}(F_{i,J}^*) \right) \\ &+ \int_I \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(f_{i,s}(U_{j,n}, V_{j,n}) - \mathbb{P}_{(U,V)}(f_{i,s}) \right) \nu_i(s) ds + o_p(1).\end{aligned}$$

We arrive at

$$(4.6) \quad \sqrt{n} \{ J_n(i) - J(i) \} = \mathbb{G}_n(F_{i,J}^*) + \int_I \mathbb{G}_n(f_{i,s}) \nu_i(s) ds + o_p(1).$$

We get the variation of J_n between to instants $i = 1$ and $i = 2$ as follows

$$\begin{aligned}\sqrt{n} \{ \Delta J_n(1, 2) - \Delta J(1, 2) \} &= \sqrt{n} \{ J_n(2) - J(2) \} - \sqrt{n} \{ J_n(1) - J(1) \} \\ &= \mathbb{G}_n (F_{2,J}^* - F_{1,J}^*) \\ &\quad + \int_I (\mathbb{G}_n(f_{2,s})\nu_2(s) - \mathbb{G}_n(f_{1,s})\nu_1(s)) ds + o_p(1).\end{aligned}$$

This leads to

$$\begin{aligned}&\sqrt{n} \{ \Delta J_n(1, 2) - \Delta J(1, 2) \} \\ &= \mathbb{G}_n (F_J^*) + \int_I (\mathbb{G}_n(f_{2,s})\nu_2(s) - \mathbb{G}_n(f_{1,s})\nu_1(s)) ds + o_p(1).\end{aligned}$$

The proof will be complete with the expression of $\Gamma_J(1, 2)$. We have

$$\begin{aligned}\Gamma_J(1, 2) &= \mathbb{E} \left(\left(\mathbb{G} (F_J^*) + \int_I (\mathbb{G}(f_{2,s})\nu_2(s) - \mathbb{G}(f_{1,s})\nu_1(s)) ds \right)^2 \right) \\ &= \mathbb{E} (\mathbb{G} (F_J^*)^2) + \mathbb{E} \left(\left(\int_I (\mathbb{G}(f_{2,s})\nu_2(s) - \mathbb{G}(f_{1,s})\nu_1(s)) ds \right)^2 \right) \\ &\quad + 2 \mathbb{E} \left(\mathbb{G} (F_J^*) \int_I (\mathbb{G}(f_{2,s})\nu_2(s) - \mathbb{G}(f_{1,s})\nu_1(s)) ds \right). \\ &\equiv \Gamma_1(1, 2) + \Gamma_2(1, 2) + 2 \Gamma_3(1, 2).\end{aligned}$$

Let us compute these three numbers. First consider,

$$\Gamma_1(1, 2) = \mathbb{E} (\mathbb{G} (F_J^*)^2) = \int_{I^2} (F_J^*(u, v) - \mathbb{P}_{(U,V)} (F_J^*))^2 dC(u, v).$$

Secondly, compute

$$\begin{aligned}\Gamma_2(1, 2) &= \mathbb{E} \left(\left(\int_I (\nu_2(s)\mathbb{G}(f_{2,s}) - \nu_1(s)\mathbb{G}(f_{1,s})) ds \right)^2 \right) \\ &= \mathbb{E} \left(\int_{I^2} [\nu_2(s)\mathbb{G}(f_{2,s}) - \nu_1(s)\mathbb{G}(f_{1,s})] [\nu_2(t)\mathbb{G}(f_{2,t}) - \nu_1(t)\mathbb{G}(f_{1,t})] ds dt \right) \\ &= \int_{I^2} \nu_2(s)\nu_2(t) \mathbb{E} (\mathbb{G}(f_{2,s}) \mathbb{G}(f_{2,t})) ds dt - \int_{I^2} \nu_2(s)\nu_1(t) \mathbb{E} (\mathbb{G}(f_{2,s}) \mathbb{G}(f_{1,t})) ds dt\end{aligned}$$

$$- \int_{I^2} \nu_1(s) \nu_2(t) \mathbb{E}(\mathbb{G}(f_{1,s}) \mathbb{G}(f_{2,t})) ds dt + \int_{I^2} \nu_1(s) \nu_1(t) \mathbb{E}(\mathbb{G}(f_{1,s}) \mathbb{G}(f_{1,t})) ds dt;$$

or

$$\mathbb{E}(\mathbb{G}(f_{2,s}) \mathbb{G}(f_{2,t})) = \mathbb{E}((\mathbb{I}_{(0,s)}(V) - s) (\mathbb{I}_{(0,t)}(V) - t)) = \min(s, t) - st;$$

$$\mathbb{E}(\mathbb{G}(f_{2,s}) \mathbb{G}(f_{1,t})) = \mathbb{E}((\mathbb{I}_{(0,s)}(V) - s) (\mathbb{I}_{(0,t)}(U) - t)) = C(t, s) - st,$$

then

$$\int_{I^2} \nu_2(s) \nu_2(t) \mathbb{E}(\mathbb{G}(f_{2,s}) \mathbb{G}(f_{2,t})) ds dt = \int_{I^2} \nu_2(s) \nu_2(t) (\min(s, t) - st) ds dt;$$

and

$$\int_{I^2} \nu_2(s) \nu_1(t) \mathbb{E}(\mathbb{G}(f_{2,s}) \mathbb{G}(f_{1,t})) ds dt = \int_{I^2} \nu_2(s) \nu_1(t) (C(t, s) - st) ds dt.$$

Similarly we obtain

$$\int_{I^2} \nu_1(s) \nu_2(t) \mathbb{E}(\mathbb{G}(f_{1,s}) \mathbb{G}(f_{2,t})) ds dt = \int_{I^2} \nu_1(s) \nu_2(t) (C(s, t) - st) ds dt;$$

$$\int_{I^2} \nu_1(s) \nu_1(t) \mathbb{E}(\mathbb{G}(f_{1,s}) \mathbb{G}(f_{1,t})) ds dt = \int_{I^2} \nu_1(s) \nu_1(t) (\min(s, t) - st) ds dt,$$

but

$$\int_{I^2} \nu_1(t) \nu_2(s) (C(t, s) - st) ds dt = \int_{I^2} \nu_1(s) \nu_2(t) (C(s, t) - st) ds dt.$$

We get identification

$$\Gamma_2(1, 2) = \gamma_1 - 2\gamma_2 + \gamma_3$$

and remind that these quantities were defined in Theorem (3). Finally, we have

$$\Gamma_3(1, 2) = \mathbb{E} \left(\mathbb{G}(F_J^*) \int_I (\mathbb{G}(f_{2,s}) \nu_2(s) - \mathbb{G}(f_{1,s}) \nu_1(s)) ds \right)$$

$$\begin{aligned}
&= \int_I \nu_2(s) \mathbb{E} (\mathbb{G}(F_J^*) \mathbb{G}(f_{2,s})) ds - \int_I \nu_1(s) \mathbb{E} (\mathbb{G}(F_J^*) \mathbb{G}(f_{1,s})) ds. \\
&= \int_I \left\{ \nu_2(s) \int_{(0,1) \times (0,s)} F_J^*(u,v) dC(u,v) - \nu_1(s) \int_{(0,s) \times (0,1)} F_J^*(u,v) dC(u,v) \right\} ds \\
&\quad - \mathbb{P}_{(U,V)}(F_J^*) \int_I s(\nu_2(s) - \nu_1(s)) ds.
\end{aligned}$$

This achieves the proof of Theorem (3).

Proof of Theorem 4.

By (4.5) and (4), is clear that the bivariate

$$(\sqrt{n}(\Delta J_n(1,2) - \Delta J(1,2)), \sqrt{n}(\Delta I_n(1,2) - \Delta I(1,2)))$$

is asymptotically Gaussian with covariance

$$\begin{aligned}
\Gamma_{I,J}(1,2) &= \mathbb{E} \left(\mathbb{G}(F_I^*) \left(\mathbb{G}(F_J^*) + \int_I (\nu_2(s) \mathbb{G}(f_{2,s}) - \nu_1(s) \mathbb{G}(f_{1,s})) ds \right) \right) \\
&= \mathbb{E} (\mathbb{G}(F_I^*) \mathbb{G}(F_J^*)) + \int_I \nu_2(s) \mathbb{E} (\mathbb{G}(F_I^*) \mathbb{G}(f_{2,s})) ds \\
&\quad - \int_I \nu_1(s) \mathbb{E} (\mathbb{G}(F_I^*) \mathbb{G}(f_{1,s})) ds.
\end{aligned}$$

Then

$$\begin{aligned}
\Gamma_{I,J}(1,2) &= \int_{I^2} (F_I^*(u,v) - \mathbb{P}_{(U,V)}(F_I^*)) (F_J^*(u,v) - \mathbb{P}_{(U,V)}(F_J^*)) dC(u,v) \\
&+ \int_I \left\{ \nu_2(s) \int_{(0,1) \times (0,s)} F_I^*(u,v) dC(u,v) - \nu_1(s) \int_{(0,s) \times (0,1)} F_I^*(u,v) dC(u,v) \right\} ds \\
&\quad - \mathbb{P}_{(U,V)}(F_I^*) \int_I s(\nu_2(s) - \nu_1(s)) ds.
\end{aligned}$$

Next straightforward computations yield

$$\begin{aligned}
\sqrt{n} \{R_n(1,2) - R(1,2)\} &= \sqrt{n} \left\{ \frac{\Delta J_n(1,2)}{\Delta I_n(1,2)} - \frac{\Delta J(1,2)}{\Delta I(1,2)} + \frac{\Delta J(1,2)}{\Delta I_n(1,2)} - \frac{\Delta J(1,2)}{\Delta I(1,2)} \right\} \\
&= \frac{1}{\Delta I_n(1,2)} \sqrt{n} \{\Delta J_n(1,2) - \Delta J(1,2)\}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\Delta J(1, 2)}{\Delta I(1, 2) \Delta I_n(1, 2)} \sqrt{n} \{ \Delta I_n(1, 2) - \Delta I(1, 2) \} \\
& = \frac{1}{\Delta I(1, 2)} \left(\mathbb{G}(F_J^*) + \int_I (\nu_2(s) \mathbb{G}(f_{2,s}) - \nu_1(s) \mathbb{G}(f_{1,s})) ds \right) \\
& \quad - \frac{\Delta J(1, 2)}{(\Delta I(1, 2))^2} \mathbb{G}(F_I^*) + o_p(1).
\end{aligned}$$

Then

$$\begin{aligned}
\sqrt{n} \{ R_n(1, 2) - R(1, 2) \} & = a \left(\mathbb{G}_n(F_J^*) + \int_I (\nu_2(s) \mathbb{G}_n(f_{2,s}) - \nu_1(s) \mathbb{G}_n(f_{1,s})) ds \right) \\
& \quad - b \mathbb{G}_n(F_I^*) + o_p(1).
\end{aligned}$$

We finish by computing its variance $\Gamma(1, 2)$. For this, let

$$\begin{aligned}
\mathbb{A}_J & = \left(\mathbb{G}(F_J^*) + \int_I (\nu_2(s) \mathbb{G}(f_{2,s}) - \nu_1(s) \mathbb{G}(f_{1,s})) ds \right), \\
\mathbb{A}_I & = \mathbb{G}(F_I^*)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(1, 2) & = \mathbb{E}((a \mathbb{A}_J - b \mathbb{A}_I)^2) \\
& = a^2 \mathbb{E}((\mathbb{A}_J)^2) + b^2 \mathbb{E}((\mathbb{A}_I)^2) - 2ab \mathbb{E}(\mathbb{A}_I \mathbb{A}_J).
\end{aligned}$$

By using the notation of Theorem 4, where we introduced a and b , we arrive at

$$\Gamma(1, 2) = a^2 \Gamma_J(1, 2) + b^2 \Gamma_I(1, 2) - 2ab \Gamma_{I,J}(1, 2).$$

This completely achieves the proofs.

5. APPENDIX AND TABLES

We use the following abbreviations in the table:

| Notations | Indices |
|--------------------------------------|--|
| $GE(\alpha)$, $\alpha = 0.5, 2, 3$ | Generalized Entropy with parameter α |
| THEIL | Theil |
| MLD | Mean Logarithmic Deviation |
| $ATK(\alpha)$, $\alpha = 0.5, -0.5$ | Atkinson with parameter α |
| CHAMP | Champernowne |
| SHOR | Shorrocks |
| SEN | Sen |
| $KAK(k)$, $k = 1, 2$ | Kakwani with parameter k |
| $FGT(\alpha)$, $\alpha = 0, 1, 2$ | Foster-Greer-Thorbecke with parameter α |

We present the results in the following tables.

| Indice I | $\Delta I(1, 2)$ | $\Gamma_I(1, 2)$ | $CI_{95\%}(\Delta I(1, 2))$ |
|-----------|------------------|------------------|------------------------------|
| GE(0.5) | -0.04025832 | 0.01770106 | $[-0.05588673, -0.03611789]$ |
| GE(2) | -0.06408679 | 0.07224733 | $[-0.09545863, -0.05552007]$ |
| GE(3) | -0.1008038 | 0.1205114 | $[-0.1495352, -0.09795348]$ |
| THEIL | -0.04569319 | 0.02223474 | $[-0.0635651, -0.04140879]$ |
| MLD | -0.03645671 | 0.01523784 | $[-0.05085476, -0.03251291]$ |
| ATK(0.5) | -0.01976068 | 0.004225092 | $[-0.02742201, -0.01776374]$ |
| ATK(-0.5) | -0.04423886 | 0.02212773 | $[-0.06159485, -0.03949192]$ |
| CHAMP | -0.03421829 | 0.01283687 | $[-0.04734396, -0.03050904]$ |

TABLE 1. Variations of the inequality indices

| Indice J | $\Delta J(1, 2)$ | $\Gamma_J(1, 2)$ | $CI_{95\%}(\Delta J(1, 2))$ |
|----------|------------------|------------------|--------------------------------|
| SHOR | -0.03024621 | 0.02353406 | $[-0.04264967, -0.01985518]$ |
| KAK(1) | -0.02108905 | 0.01097123 | $[-0.02982085, -0.01425729]$ |
| KAK(2) | -0.02055594 | 0.01007820 | $[-0.02961271, -0.01469601]$ |
| FGT(0) | -0.05977098 | 0.3170756 | $[-0.09355847, -0.009889805]$ |
| FGT(1) | -0.01859332 | 0.00922992 | $[-0.02620413, -0.01192899]$ |
| FGT(2) | -0.00432289 | 0.0008381113 | $[-0.007194404, -0.002892781]$ |

TABLE 2. Variations of the povrety indices

| Ratio | $R(1, 2)$ | $\Gamma_{IJ}(1, 2)$ | $\Gamma(1, 2)$ | $CI_{95\%}(R(1, 2))$ |
|------------------|-----------|---------------------|----------------|------------------------|
| SHOR/GE(0.5) | 0.7513034 | 0.005477263 | 15.60737 | [0.3858608, 0.9728719] |
| SHOR/GE(2) | 0.471957 | 0.006487665 | 8.157275 | [0.2018082, 0.6261873] |
| SHOR/GE(3) | 0.3000503 | 0.009018111 | 2.851175 | [0.1271085, 0.3780043] |
| SHOR/THEIL | 0.6619413 | 0.005642781 | 12.36007 | [0.3342390, 0.8566255] |
| SHOR/MLD | 0.8296473 | 0.8296473 | 18.77303 | [0.4278509, 1.071647] |
| SHOR/ATKIN(0.5) | 1.530626 | 0.002695030 | 64.49043 | [0.7866646, 1.979908] |
| SHOR/ATKIN(-0.5) | 0.6837023 | 0.007288597 | 12.21780 | [0.555278, 1.395697] |
| SHOR/CHAMP | 0.8839194 | 0.005165236 | 20.86647 | [0.4634852, 1.142229] |

TABLE 3. Ratio of the variations with Shorrocks

| Ratio | $R(1, 2)$ | $\Gamma_{IJ}(1, 2)$ | $\Gamma(1, 2)$ | $CI_{95\%}(R(1, 2))$ |
|---------------|-----------|---------------------|----------------|------------------------|
| SEN/GE(0.5) | 0.3290702 | 0.003112166 | 7.754599 | [0.272201, 0.6859714] |
| SEN/GE(2) | 0.3290702 | 0.003512353 | 4.013294 | [0.1431155, 0.4407834] |
| SEN/GE(3) | 0.2092089 | 0.005939808 | 1.354192 | [0.0916464, 0.2645570] |
| SEN/THEIL | 0.461536 | 0.003364929 | 6.035583 | [0.237376, 0.6024165] |
| SEN/MLD | 0.5784683 | 0.002968939 | 9.506736 | [0.2996504, 0.7577893] |
| SEN/ATK(0.5) | 1.067223 | 0.001542060 | 31.99108 | [0.555278, 1.395697] |
| SEN/ATK(-0.5) | 0.4360427 | 0.003368434 | 6.534366 | [0.2461303, 0.625955] |
| SEN/CHAMP | 0.6163094 | 0.003038844 | 10.33521 | [0.3273292, 0.8050137] |

TABLE 4. Ratio of the variations with Sen

| Ratio | $R(1, 2)$ | $\Gamma_{IJ}(1, 2)$ | $\Gamma(1, 2)$ | $CI_{95\%}(R(1, 2))$ |
|------------------|-----------|---------------------|----------------|-------------------------|
| KAK(2)/GE(0.5) | 0.510601 | 0.002574653 | 7.443462 | [0.2788993, 0.6842854] |
| KAK(2)/GE(2) | 0.3207516 | 0.008486058 | 2.93814 | [0.1661299, 0.4208233] |
| KAK(2)/GE(3) | 0.2039203 | 0.005185377 | 1.276858 | [0.09508295, 0.2629838] |
| KAK(2)/THEIL | 0.4498688 | 0.002906321 | 5.72986 | [0.2442552, 0.5999303] |
| KAK(2)/MLD | 0.5638451 | 0.002365820 | 9.220372 | [0.3058926, 0.7570787] |
| KAK(2)/ATK(0.5) | 1.040245 | 0.001292464 | 30.63183 | [0.5694048, 1.391776] |
| KAK(2)/ATK(-0.5) | 0.4646579 | 0.001933209 | 6.672792 | [0.2464103, 0.630237] |
| KAK(2)/CHAMP | 0.6007296 | 0.002781442 | 9.709634 | [0.3376321, 0.8006341] |

TABLE 5. Ratio of the variations with Kakwani

| Ratio | $R(1, 2)$ | $\Gamma_{IJ}(1, 2)$ | $\Gamma(1, 2)$ | $CI_{95\%}(R(1, 2))$ |
|------------------|-----------|---------------------|----------------|---------------------------|
| FGT(0)/GE(0.5) | 1.484686 | 1.484686 | 192.9616 | [0.09236428, 2.156398] |
| FGT(0)/GE(2) | 0.9326567 | 0.02159780 | 82.69382 | [0.009587167, 1.360782] |
| FGT(0)/GE(3) | 0.5929437 | 0.03215672 | 31.62072 | [0.0002219161, 0.8357621] |
| FGT(0)/THEIL | 1.308094 | 0.01626234 | 149.7108 | [0.07643712, 1.894496] |
| FGT(0)/MLD | 1.639505 | 0.01332770 | 236.7108 | [0.09833456, 2.383401] |
| FGT(0)/ATK(0.5) | 3.024743 | 0.00717539 | 799.837 | [0.1882737, 4.390527] |
| FGT(0)/ATK(-0.5) | 1.351097 | 0.01606948 | 160.4669 | [0.08224307, 1.964480] |
| FGT(0)/CHAMP | 1.746755 | 0.01248913 | 266.9863 | [0.1148277, 2.542700] |

TABLE 6. Ratio of the variations with FGT(0)

| Ratio | $R(1, 2)$ | $\Gamma_{IJ}(1, 2)$ | $\Gamma(1, 2)$ | $CI_{95\%}(R(1, 2))$ |
|------------------|------------|---------------------|----------------|------------------------|
| FGT(1)/GE(0.5) | 0.4618504 | 0.003359959 | 6.109622 | [0.2308332, 0.5981059] |
| FGT(1)/GE(2) | 0.29901272 | 0.004159761 | 3.140289 | [0.2316082, 0.4949175] |
| FGT(1)/GE(3) | 0.1844506 | 0.005815332 | 1.100702 | [0.0761356, 0.2320249] |
| FGT(1)/THEIL | 0.4069167 | 0.003487018 | 4.824886 | [0.2000723, 0.5264534] |
| FGT(1)/MLD | 0.5100109 | 0.003329621 | 7.371324 | [0.2557003, 0.6591174] |
| FGT(1)/ATK(0.5) | 0.9409253 | 0.001652060 | 25.25488 | [0.4705622, 1.217276] |
| FGT(1)/ATK(-0.5) | 0.4202938 | 0.004429351 | 4.81098 | [0.2142764, 0.5401868] |
| FGT(1)/CHAMP | 0.5433737 | 0.003126249 | 8.218207 | [0.2768286, 0.7027897] |

TABLE 7. Ratio of the variations with FGT(1)

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| Ratio | $R(1, 2)$ | $\Gamma_{IJ}(1, 2)$ | $\Gamma(1, 2)$ | $CI_{95\%}(R(1, 2))$ |
|------------------|------------|---------------------|----------------|--------------------------|
| FGT(2)/GE(0.5) | 0.1073788 | 0.000974483 | 0.5139224 | [0.05637792, 0.1628977] |
| FGT(2)/GE(2) | 0.06745369 | 0.001055690 | 0.2494247 | [0.02970793, 0.103916] |
| FGT(2)/GE(3) | 0.0428842 | 0.001371335 | 0.09271563 | [0.01813633, 0.06338001] |
| FGT(2)/THEIL | 0.09460689 | 0.0009653898 | 0.4092489 | [0.04856479, 0.1436198] |
| FGT(2)/MLD | 0.118576 | 0.001013111 | 0.6110173 | [0.06292282, 0.1790699] |
| FGT(2)/ATK(0.5) | 0.2187623 | 0.0004795731 | 2.126811 | [0.1148914, 0.3315849] |
| FGT(2)/ATK(-0.5) | 0.09771703 | 0.001424631 | 0.3939442 | [0.05315702, 0.1464178] |
| FGT(2)/CHAMP | 0.1263327 | 0.000954164 | 0.6848654 | [0.0680842, 0.1910499] |

TABLE 8. Ratio of the variations with FGT(2)

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